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# HELMHOLTZ'S THIRD THEOREM IN NUMERICAL GENERAL RELATIVITY

APS April Meeting, Columbus, OH, 17 April 2018

# NUMERICAL RELATIVITY

- Gravitational waves from compact binaries carry unique information on their properties and probe physics inaccessible to telescopes or laboratories. Binary neutron stars are astrophysical scale particle colliders.
- Although development of black-hole gravitational wave templates in the past decade has been revolutionary, the corresponding work for double neutron-star systems has lagged.
- Numerical relativity is absolutely crucial for the development of gravitational wave templates for NS-NS and BH-NS binaries.
- The Valencia scheme has been a workhorse for hydrodynamics in numerical relativity…

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- Numerical relativity is absolutely crucial for the development of gravitational wave templates for NS-NS and BH-NS binaries.
- The Valencia scheme has been a workhorse for hydrodynamics in numerical relativity…

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$\nabla_\alpha(\rho u^\alpha) = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,\rho u^\alpha) = 0$$

$$\nabla_\beta T_\alpha{}^\beta = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}\,T_\alpha{}^\beta) - \Gamma^\gamma_{\alpha\beta} T_\gamma{}^\beta = 0$$

# FLUID DYNAMICS IN NUMERICAL RELATIVITY

- **Valencia formulation**:
  + Flux-conservation form, standard shock-capturing schemes applicable
  – Needs 'conservative' to 'primitive' routine, ill-posed on vacuum, needs artificial atmosphere
- **Walton-Fraundiener formulation**:
  Symmetric hyperbolic
- **Hamiltonian formulation**:
  Canonical methods have influenced all areas of physics
  Application in fluids (Synge, Lichnerowicz, Carter, Markakis et al.) very promising

B. Carter, *Perfect fluid and magnetic field conservation laws in the theory of black hole accretion rings*, in Active Galactic Nuclei, 273-300, 1979
A. Walton, Houston J. Math., 31, 145-160, 2005, J. Frauendiener, CQG 20, L193–L196, 2003
C. Markakis, arXiv:1410.7777, C. Markakis et al. arXiv:1612.09308

# ACOUSTICAL & CANONICAL FLUID DYNAMICS

$$\begin{cases} \partial_t \rho = \ldots \\ \partial_t (\rho u_i) = \ldots \end{cases} \rightarrow \begin{cases} \partial_t H = \ldots \\ \partial_t p_i = \ldots \end{cases}$$

# ACTION PRINCIPLE FOR FLUIDS

- Carter–Lichnerowicz described barotropic fluid motion as **conformally geodesic**

$$S = \int_{\mathrm{i}}^{\mathrm{f}} h \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}\, dt \qquad\qquad h = 1 + \int_0^\rho \frac{dp}{\rho}$$

- Helmholtz's 3rd theorem: ***initially irrotational flows remain irrotational***

- These concepts lead to **Lagrangian**, **Hamiltonian** or **Hamilton-Jacobi** schemes, with novel applications in numerical relativity, BNS inspiral, and fluid dynamics

# ACTION PRINCIPLE FOR FLUIDS

"He got the action, he got the motion" – Dire Straights

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Euler-Lagrange: $$\frac{dp_a}{dt} - \frac{\partial L}{\partial x^a} = (\partial_t + \pounds_v)p_a - \nabla_a L = 0$$

Hamilton: $$\frac{dp_a}{dt} + \frac{\partial H}{\partial x^a} = \partial_t p_a + v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a H = 0$$

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The above equations are covariant and valid in both Newtonian and relativistic contexts

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Constrained Hamiltonian: $H = p_a v^a - L = -p_a \beta^a + \alpha \sqrt{h^2 + \gamma^{ab} p_a p_b}$

# ACTION PRINCIPLE FOR FLUIDS (NEWTONIAN)

$$S = \int_{\mathrm{i}}^{\mathrm{f}} \left(\tfrac{1}{2}\gamma_{ab} v^a v^b - h - \Phi\right) dt$$

Euler-Lagrange: $\dfrac{dp_a}{dt} - \dfrac{\partial L}{\partial x^a} = (\partial_t + \pounds_v) p_a - \nabla_a L = 0$

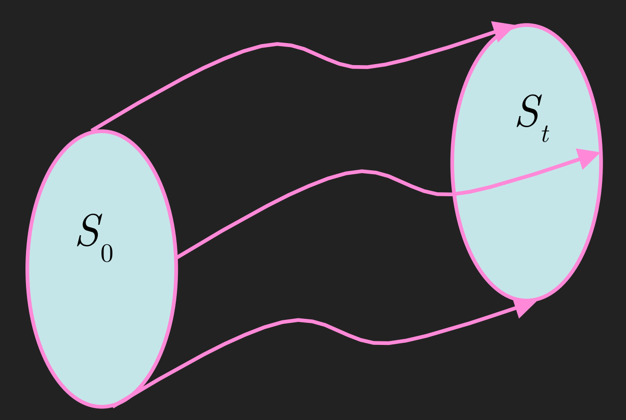
Hamilton: $\dfrac{dp_a}{dt} + \dfrac{\partial H}{\partial x^a} = \partial_t p_a + v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a H = 0$

Hamiltonian: $H = p_a v^a - L = \tfrac{1}{2}\gamma^{ab} p_a p_b + h + \Phi$

# CONSERVATION OF CIRCULATION

- A system is Hamiltonian iff it possesses a Poincaré-Cartan integral invariant

$$\oint_{\partial S_t} \mathbf{p} = \int_{S_t} \mathbf{dp} = \text{constant}$$

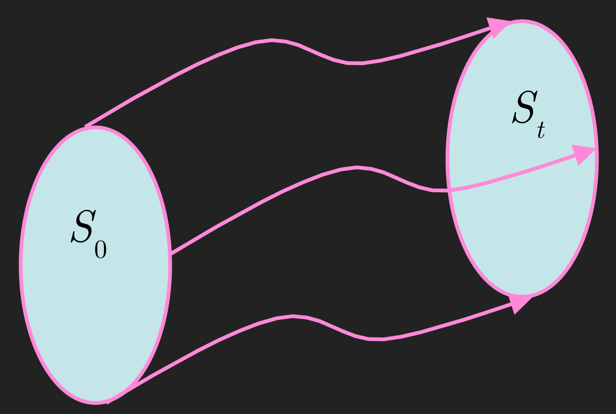



- Kelvin’s theorem a special case of this invariant (generalizable)
- Kelvin's theorem is **exact in time-dependent spacetimes, with gravitational waves carrying energy and angular momentum away from a system**
- **Corollary**: flows initially irrotational remain irrotational **(Helmholtz’s third theorem)**

C. Markakis et al., *Conservation laws and evolution schemes in geodesic, hydrodynamic and magnetohydrodynamic flows*, PRD 96 064019 (2017)

# CONSERVATION OF CIRCULATION

- Kelvin: circulation theorem for *barotropic* (isentropic or cold) fluids




- Ertel: potential vorticity theorem for *baroclinic* (non-isentropic) fluids
- Carter: circulation theorem for *barotropic*, poorly conducting, magnetofluids
- Bekenstein-Oron: circulation theorem for *barotropic*, perfectly conducting, magnetofluids
- Markakis et al.: Generalized Ertel's theorem to *any* Hamiltonian system
  Generalized Kelvin's theorem to *baroclinic*, poorly or perfectly conducting, magnetofluids

  C. Markakis et al., *Conservation laws and evolution schemes in geodesic, hydrodynamic and magnetohydrodynamic flows*, PRD 96 064019 (2017)

# CANONICAL FLUID DYNAMICS

Irrotational flow: $p_a = \nabla_a S$

Hamiltonian: $H = -p_a \beta^a + \alpha\sqrt{h^2 + \gamma^{ab} p_a p_b}$

Hamilton-Jacobi: $\partial_t S + H = 0$

Example: Pressureless 'dust' on Minkowski spacetime yields relativistic Burgers equation:

$$\partial_t S + \sqrt{1 + (\nabla S)^2} = 0$$

LeFloch, Makhlofand and Okutmustur, SINUM 50, 2136 (2012) used noncovariant algebraic manipulation of the Euler eqs., repeated on each chart. Their covariant, Hamiltonian form went unnoticed.

Solutions to HJ equation are NOT unique.
Nevertheless, 'viscosity' solutions to HJ equation are unique.

C. Markakis, arXiv:1410.7777, C. Markakis et al. arXiv:1612.09308

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AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, July 1992, Pages 1-67

# USER'S GUIDE TO VISCOSITY SOLUTIONS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

MICHAEL G. CRANDALL, HITOSHI ISHII, AND PIERRE-LOUIS LIONS

ABSTRACT. The notion of viscosity solutions of scalar fully nonlinear partial differential equations of second order provides a framework in which startling comparison and uniqueness theorems, existence theorems, and theorems about continuous dependence may now be proved by very efficient and striking arguments. The range of important applications of these results is enormous. This article is a self-contained exposition of the basic theory of viscosity solutions.

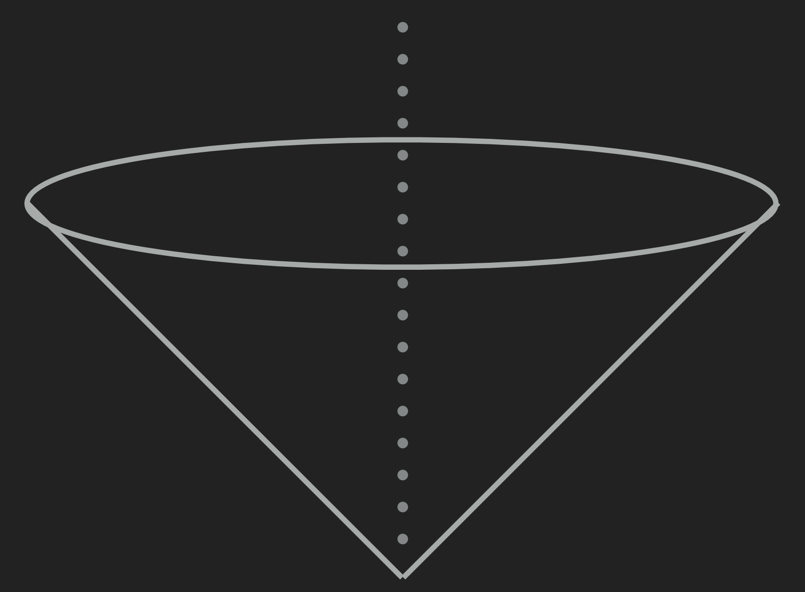
# ACOUSTICAL FLUID DYNAMICS

Euler's equation is coupled to the continuity equation

$$a^{\mu\nu}\nabla_\mu p_\nu = 0$$

where $a^{\mu\nu}$ is the inverse of the acoustic metric:

$$a_{\mu\nu}dx^\mu dx^\nu = -c_s^{\,2}dt^2 + \gamma_{ij}(dx^i - v^i dt)(dx^i - v^j dt)$$



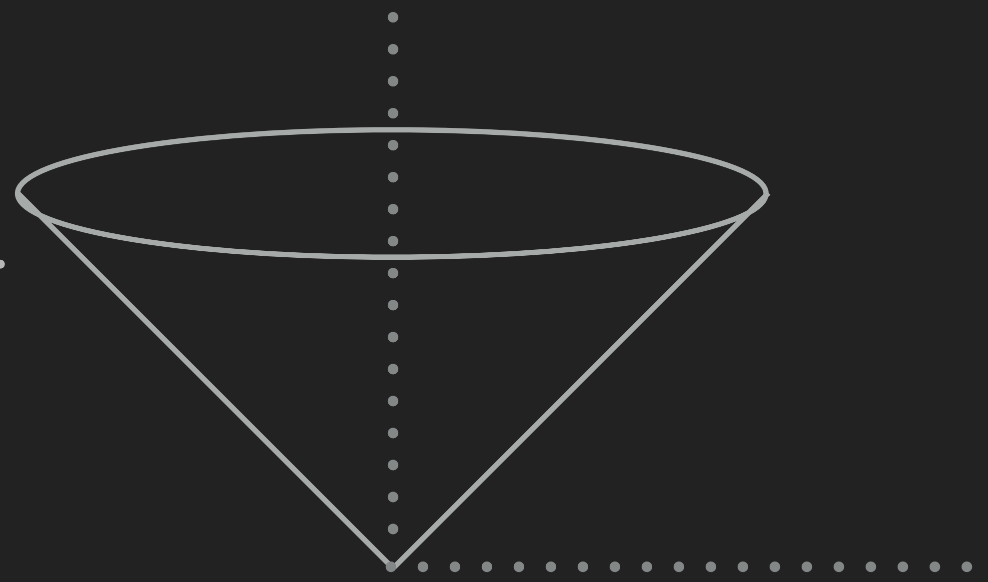
The null cones of the acoustic metric are the **sound cones**.

# ACOUSTICAL FLUID DYNAMICS

Euler's equation is coupled to the continuity equation. For irrotational flow, they amount to a non-linear Klein-Gordon eq.:

$$a^{\mu\nu}\nabla_\mu \nabla_\nu S = 0$$

Stiff fluid, $c_s = 1,\ a_{\mu\nu} = g_{\mu\nu}$, reduces to linear massless Klein-Gordon eq. Codes that can evolve scalar fields, can evolve fluids with the same infrastructure.



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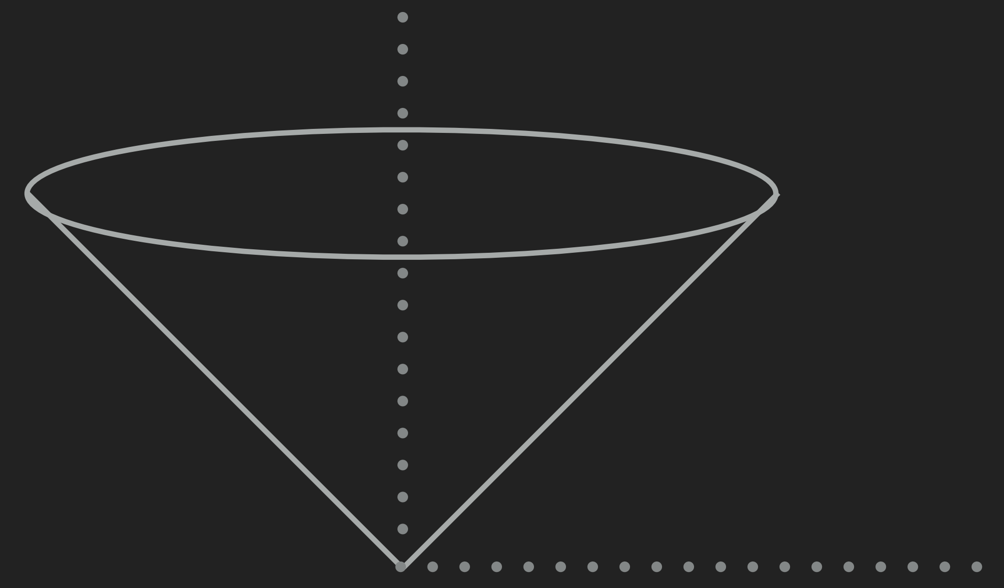

The null cones of the acoustic metric are the **sound cones**.

$$\text{Characteristics}: \quad \lambda^{k}_{1,2} = 0 \text{ or } v^{k}, \ \lambda^{k}_{3,4} = v^{k} \pm c_{\mathrm{s}}$$

Strongly hyperbolic (**well posed**) for $c_{\mathrm{s}} > 0$

C. Markakis, arXiv:1410.7777, C. Markakis et al. arXiv:1612.09308

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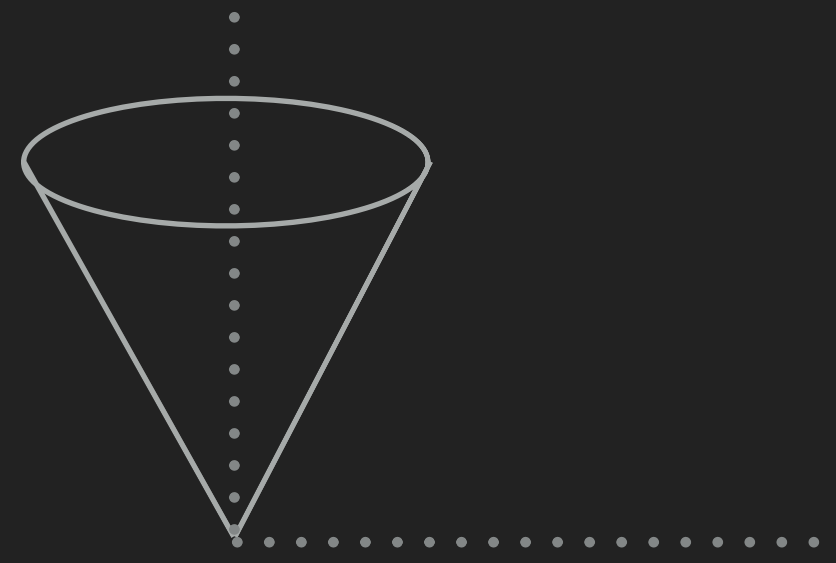

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The null cones of the acoustic metric are the **sound cones**.

Characteristics : $\lambda^{k}_{1,2} = 0$ or $v^{k}$, $\lambda^{k}_{3,4} = v^{x} \pm c_{s}$

Not hyperbolic (**ill posed**) for $c_{s} = 0$

# ACOUSTICAL FLUID DYNAMICS

$$\begin{cases} \partial_t S + H - \gamma_1 N^i (\partial_i S - p_i) \\ (\partial_t - 2N^i \partial_i) H + (N^2 \Gamma^{ij} - N^i N^j) \partial_i p_j + \Delta - \gamma_1 \gamma_2 N^i (\partial_i S - p_i) = 0 \\ \partial_t p_i + \partial_i H - \gamma_2 N (\partial_i S - p_i) = 0 \end{cases}$$

$$N = c_{\mathrm{s}},\ N^i = -v^i,\ \Gamma^{ij} = \gamma^{ij} \quad \text{(Newtonian)}$$

$$N = \alpha v_{\mathrm{s}},\ N^i = -(1 - v_{\mathrm{s}}^2) v^j + v_{\mathrm{s}}^2 \beta^j,\ \Gamma^{ij} = \gamma^{ij} - (1 - v_{\mathrm{s}}^2) \nu^i \nu^j \quad \text{(GR)}$$

$$v_{\mathrm{s}} := c_{\mathrm{s}} [1 + (1 - c_{\mathrm{s}}^2) \gamma^{kl} u_k u_l]^{-1/2}$$

Stiff fluid: $c_{\mathrm{s}} = 1$

The difficulty and importance of the breakdown of Euler's equations on a vacuum boundary was discussed by von Neumann, Heisenberg, Burgers and other participants of a conference, *Problems in cosmical aerodynamics*, held in Paris 1949. It is still an open problem today.

A discussion of the session may be found in: Juhi Jang, Nader Masmoudi, *Well-posedness of compressible Euler equations in a physical vacuum* [arXiv:1005.4441]

# COMMENTS

- Like Valencia, hyperbolicity lost when $c_s = 0$, causing instability
- Unlike Valencia, requires no atmosphere. Direct control over acoustic structure can be used to **regularize Euler equations** on vacuum boundary and enforce **Helmholtz's 3rd theorem**
- All known classical mechanics methods (e.g. symplectic integrators) applicable
- Shocks treatable with Cole-Hopf transformation
- Extension beyond irrotational flows possible
- Code part of the open source 'Einstein Toolkit'

C. Markakis, *Hamiltonian Hydrodynamics and irrotational binary inspiral*, arXiv:1410.7777
C. Markakis et al., *Conservation laws and evolution schemes in geodesic, hydrodynamic and magnetohydrodynamic flows*, arXiv:1612.09308